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A Hilbert transform representation of the error in Lagrange interpolation

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Abstract

Let $L_n[f]$ denote the Lagrange interpolation polynomial to a function f at the zeros of a polynomial P_n with distinct real zeros. We show that

$$f - L_n[f] = -P_n H_e \left[\frac{H[f]}{P_n} \right],$$

where H denotes the Hilbert transform, and H_e is an extension of it. We use this to prove convergence of Lagrange interpolation for certain functions analytic in $(-1, 1)$ that are not assumed analytic in any ellipse with foci at $(-1, 1)$.

Keywords: Lagrange interpolation; Hilbert tranform

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1. Introduction and results

Let P_n be a polynomial of degree n with distinct real zeros, and given a function f defined at least on these zeros, let $L_n[f]$ denote the Lagrange interpolation polynomial to f at the zeros of P_n. Analysis of the error $f - L_n[f]$ depends on a

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suitable representation of it [1,4,6]. For functions analytic in a simply connected set containing the zeros of P_n, one can use Hermite's contour integral error formula. For functions with sufficiently many derivatives, one can use integral forms of the remainder. The latter may even be formulated for functions without derivatives in terms of divided differences. When the interpolation points are zeros of orthogonal polynomials, one can use special identities [3].

In this note, we present a representation for the error involving the Hilbert transform. As far as we can determine it is new, although for a very long time the Hilbert transform has been used in studying Lagrange interpolation (for example, see [3]). Then we use this to study convergence of Lagrange interpolation for functions whose Hilbert transform vanishes in the interval, say $(-1,1)$, containing the interpolation points. This forces analyticity of the function in most of the plane. However, it does allow functions that are not analytic in an ellipse with foci at $(-1,1)$—the traditional hypothesis in studying Lagrange interpolation of analytic functions, when the interpolation points lie in $(-1,1)$.

Given a function $f \in L_1(\mathbb{R})$, its Hilbert transform is defined for a.e. $x \in \mathbb{R}$ by

$$H[f](x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(s)}{s-x} ds.$$

Here PV denotes Cauchy principal value. The Hilbert transform is a bounded operator on $L_p(\mathbb{R})$, if $p > 1$. That is, there exists C_p depending only on p such that for all $f \in L_p(\mathbb{R})$,

$$||H[f]||_{L_p(\mathbb{R})} \leqslant C_p \|f\|_{L_p(\mathbb{R})}. \tag{1}$$

Moreover, $-H \circ H$ is the identity. That is, if $p > 1$ and $f \in L_p(\mathbb{R})$, then for a.e. x,

$$H \circ H[f](x) = -f(x). \tag{2}$$

See for example [5, Chapter 5]. When f has finitely many non-integrable singularities, say at $a_1, a_2, \ldots, a_m$, but is integrable in $\mathbb{R} \setminus \bigcup_{j=1}^{m} (a_j - \varepsilon, a_j + \varepsilon)$ for each $\varepsilon > 0$, we extend the definition of H as a principal value integral. Set $a_0 = x$ and if $x \notin \{a_1, a_2, \ldots, a_m\}$, define

$$H_{\mathrm{e}}[f](x) = \frac{1}{\pi} \lim_{\varepsilon_j \to 0+} \int_{\mathbb{R} \setminus \bigcup\limits_{j=0}^{m} [a_j - \varepsilon_j, a_j + \varepsilon_j]} \frac{f(s)}{s-x} ds,$$

where the limit is taken as each $\varepsilon_j \to 0+$, $0 \leqslant j \leqslant m$, independently. If this limit exists, the extended transform is well defined at x. With this extension, we prove:

Theorem 1. *Let $n \geqslant 1$, and P_n be a polynomial of degree n with n distinct real zeros. Let $p > 1$ and let $f \in L_p(\mathbb{R})$. Assume moreover that the inversion formula* (2) *is valid at every zero of P_n. Let U be a polynomial of degree at most n and S be a polynomial of degree at most $n-1$. Then for a.e. x,*

$$Uf - L_n[Uf] = -P_n H_{\mathrm{e}} \left[\frac{UH[f] - S}{P_n} \right]. \tag{3}$$

Remarks. (a) Note that since $f \in L_p(\mathbb{R})$, also $H[f] \in L_p(\mathbb{R})$. Then the inversion formula (2) is valid a.e. Our hypothesis is that (2) holds at each zero x of P_n. If in addition, f satisfies a Lipschitz condition of some positive order in a neighbourhood of each of the zeros of P_n, Privalov's theorem shows that the same is true of $H[f]$. Then the inversion formula (2) holds pointwise in a neighbourhood of each of the zeros of P_n, so (3) does also. In particular, if f satisfies a local Lipschitz condition everywhere in $\mathbb{R}$, (3) holds except at the zeros of P_n.

(b) We can weaken the requirement on f: it suffices that $f \in L\log^+ L(\mathbb{R})$ for $H[f] \in L_1(\mathbb{R})$.

(c) When $U \equiv 1$ and $S \equiv 0$, we obtain

$$f - L_n[f] = -P_n H_{\mathrm{e}}\left[\frac{H[f]}{P_n}\right] \tag{4}$$

and hence

$$\begin{aligned} L_n[f](x) &= f(x) + P_n(x) H_{\mathrm{e}}\left[\frac{H[f]}{P_n}\right](x) \\ &= H_{\mathrm{e}}\left[H[f]\left(\frac{P_n(x)}{P_n} - 1\right)\right](x), \end{aligned}$$

in view of (2). Of course, $P_n(x)$ is regarded as constant inside the Hilbert transform.

(d) The idea for the proof comes essentially from [2], where a new representation was established for the error in Lagrange interpolation of $x^\alpha, \alpha > 0$. The new twist in this paper over [2] is the use of singular integrals and invertibility of the Hilbert transform.

Corollary 2. *Let I be a real interval and $W : I \to \mathbb{R}$ be measurable. Let $1 < p < \infty$, and $r, s \geqslant 1$ with $\frac{1}{r} + \frac{1}{s} = 1$. Let S be a polynomial of degree $\leqslant n - 1$. Then provided $WP_n \in L_{pr}(I)$ and $(H[f] - S)/P_n \in L_{ps}(\mathbb{R})$,*

$$||W(f - L_n[f])||_{L_p(I)} \leqslant C_{ps} ||WP_n||_{L_{pr}(I)} \left|\left|\frac{H[f] - S}{P_n}\right|\right|_{L_{ps}(\mathbb{R})}, \tag{5}$$

where C_{ps} depends only on ps.

Remarks. (a) In particular, if f satisfies a Lipschitz condition of positive order near each zero of P_n, we see that $H[f] - L_n[H[f]]$ satisfies a Lipschitz condition near each of the zeros of P_n, so

$$||W(f - L_n[f])||_{L_p(I)} \leqslant C_{ps} ||WP_n||_{L_{pr}(I)} \left|\left|\frac{H[f] - L_n[H[f]]}{P_n}\right|\right|_{L_{ps}(\mathbb{R})},$$

a curious duality result.

(b) Of course the real restriction is that $(H[f] - S)/P_n \in L_{ps}(\mathbb{R})$. Here $1/P_n$ has non-integrable singularities at the zeros of P_n, but we can satisfy this by requiring that $H[f] - S$ vanishes in a neighbourhood of the zeros of P_n—for example in an interval I containing the zeros of P_n. This forces analyticity of f in $(\mathbb{C}\backslash\mathbb{R})\cup I$, and explains the hypotheses in the following theorem:

Theorem 3. *For* $n\geqslant 1$, *let* P_n *be a polynomial of degree* n *with* n *distinct zeros in* $(-1,1)$. *Let* $1<p<\infty$ *and* $q=\frac{p}{p-1}$. *Assume that for each* $0<\varepsilon<1$,

$$\lim_{n\to\infty} ||P_n||_{L_\infty[-1+\varepsilon,1-\varepsilon]}\left|\left|\frac{1}{P_n}\right|\right|_{L_q(\mathbb{R}\backslash[-1,1])}=0. \tag{6}$$

Let $f:(-1,1)\to\mathbb{R}$ *be the restriction to* $(-1,1)$ *of a function analytic in* $\mathbb{C}\backslash[-1,1]$, *with boundary values a.e. on* $\mathbb{R}\backslash[-1,1]$, *from the upper and lower half-planes, that lie in* $L_q(\mathbb{R}\backslash[-1,1])$. *Assume moreover, that* f *has limit* 0 *at* ∞. *Then for each* $\varepsilon>0$,

$$\lim_{n\to\infty} ||f-L_n[f]||_{L_\infty[-1+\varepsilon,1-\varepsilon]}=0. \tag{7}$$

Remarks. (a) When discussing convergence of Lagrange interpolation for an array of interpolation points in $(-1,1)$ and for functions analytic there, one invariably assumes the function is analytic in a neighbourhood of $[-1,1]$—typically an ellipse with foci at ± 1. Theorem 3 allows functions that are not analytic in a neighbourhood of $[-1,1]$—for example,

$$f(x)=(1-x^2)^{-\alpha},\quad x\in(-1,1),\quad 0<\alpha<1.$$

(b) Chebyshev polynomials—and more generally Jacobi polynomials—satisfy (6).

We prove the theorems in the next section.

2. Proofs

We begin with

Proof of Theorem 1. Let $s\in\mathbb{R}$ with $P_n(s)\neq 0$ and let

$$h_s(x)=\frac{1}{s-x},\quad x\in\mathbb{R}\backslash\{s\}.$$

Then $L_n[Uh_s]$ is a well-defined polynomial of degree $\leqslant n-1$ that agrees with Uh_s at the zeros of P_n. It follows that $U-L_n[Uh_s]/h_s$ is a polynomial of degree $\leqslant n$ that vanishes at the zeros of P_n. Then for some constant c,

$$U-L_n[Uh_s]/h_s=cP_n.$$

Evaluating both sides at s gives

$$c=U(s)/P_n(s).$$

So for $x \neq s$,

$$\frac{U(x)}{s-x} - L_n[Uh_s](x) = \frac{U(s)\ P_n(x)}{P_n(s)(s-x)} \quad . \tag{8}$$

Now, we let

$$g = H[f].$$

Our hypotheses on f ensure that $g \in L_p(\mathbb{R})$ and that g is defined a.e. in $\mathbb{R}$. Multiplying (8) by $\frac{1}{\pi} g(s)$ and integrating in a principal value sense with respect to s over $\mathbb{R}$, gives for a.e. x,

$$U(x)H[g](x) - L_n[UH[g]](x) = P_n(x)H_{\mathrm{e}}\left[\frac{Ug}{P_n}\right](x).$$

Note that the interchange of L_n and H on the left is permissible as $L_n[Uh_s]$ may be expressed as a finite linear combination of $\frac{1}{s-x_j}$, where $x_1, x_2, \ldots, x_n$ are the zeros of P_n. Then the right-hand side will be well defined in the sense of the extended definition of the Hilbert transform given in the introduction. Since the limiting process defining H_{e} gives a finite limit on the left a.e., the same will be true for the right-hand side. Recalling that a.e.

$$H[g] = H \circ H[f] = -f$$

and that this holds by hypothesis at the zeros of P_n, we obtain for a.e. x,

$$U(x)f(x) - L_n[Uf](x) = -P_n(x)H_{\mathrm{e}}\left[\frac{UH[f]}{P_n}\right](x). \tag{9}$$

Then (3) will follow if we show that for every polynomial S of degree $\leqslant n-1$,

$$H_{\mathrm{e}}\left[\frac{S}{P_n}\right] = 0,$$

except possibly at the zeros of P_n. Since S/P_n is a linear combination of $h_{x_j}, j = 1, 2, \ldots, n$, it suffices to show that

$$H_{\mathrm{e}}[h_a](x) = 0, \quad x \in \mathbb{R}\backslash\{a\}. \tag{10}$$

But

$$H_{\mathrm{e}}[h_a](x)$$
$$= \frac{1}{\pi} \lim_{\varepsilon_j \to 0} \int_{\mathbb{R}\backslash((x-\varepsilon_0, x+\varepsilon_0) \cup (a-\varepsilon_1, a+\varepsilon_1))} \frac{1}{(s-x)(s-a)} ds$$

$$=\frac{1}{\pi}\frac{1}{x-a}\lim_{\varepsilon_j\to 0}\int_{\mathbb{R}\setminus((x-\varepsilon_0,x+\varepsilon_0)\cup(a-\varepsilon_1,a+\varepsilon_1))}\left[\frac{1}{s-x}-\frac{1}{s-a}\right]ds$$

$$=\frac{1}{\pi}\frac{1}{x-a}\lim_{\varepsilon_j\to 0,R\to\infty}\left(\int_{[-R,R]\setminus((x-\varepsilon_0,x+\varepsilon_0)\cup(a-\varepsilon_1,a+\varepsilon_1))}\left[\frac{1}{s-x}-\frac{1}{s-a}\right]ds\right.$$
$$\left.+O\left(\frac{1}{R}\right)\right)$$

$$=\frac{1}{\pi}\frac{1}{x-a}\lim_{R\to\infty}\left(\log\left|\frac{R-x}{R+x}\right|-\log\left|\frac{R-a}{R+a}\right|\right)=0.$$

So we have (10) and the result. □

Proof of Corollary 2. Our hypothesis that $(H[f]-S)/P_n\in L_{ps}(\mathbb{R})$ reduces the extended Hilbert transform to an ordinary one

$$H_{\mathrm{e}}\left[\frac{H[f]-S}{P_n}\right]=H\left[\frac{H[f]-S}{P_n}\right].$$

By Hölder's inequality, and then boundedness of the Hilbert transform on $L_{ps}(\mathbb{R})$,

$$\|W(f-L_n[f])\|_{L_p(I)}\leqslant\|WP_n\|_{L_{pr}(I)}\left\|H\left[\frac{H[f]-S}{P_n}\right]\right\|_{L_{ps}(I)}$$
$$\leqslant C_{ps}\|WP_n\|_{L_{pr}(I)}\left\|\frac{H[f]-S}{P_n}\right\|_{L_{ps}(\mathbb{R})},$$

where C_{ps} is the norm of the Hilbert transform as an operator from $L_{ps}(\mathbb{R})$ to $L_{ps}(\mathbb{R})$. □

Proof of Theorem 3. Let $z\in\mathbb{C}\setminus[-1,1]$. Let Γ be a simple closed positively oriented contour in $\mathbb{C}\setminus((-\infty,-1]\cup[1,\infty))$ enclosing z. We have

$$f(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(s)}{s-z}ds.$$

By deforming Γ onto $(-\infty,-1]\cup[1,\infty)$, and using that f has limit 0 at ∞, we obtain

$$f(z)=\frac{1}{2\pi i}\left[\int_{-\infty}^{-1}+\int_1^{\infty}\right]\frac{f(s+)-f(s-)}{s-z}ds,$$

where $f(s\pm)$ denote boundary values from the upper and lower half-planes, respectively. Let

$$g(s)=\begin{cases}\frac{1}{2i}[f(s+)-f(s-)], & s\in\mathbb{R}\setminus[-1,1],\\ 0, & s\in(-1,1).\end{cases}$$

By hypothesis $g\in L_p(\mathbb{R})$ and we see that

$$f=H[g]\quad\text{in }(-1,1).$$

We extend f to $\mathbb{R}\backslash[-1,1]$ by defining

$$f = H[g]$$

there. (Equivalently, the Sokotkii–Plemelj formulas show that we can define f as the average of its boundary values from the upper and lower half-planes there

$$f(s) = \tfrac{1}{2}(f(s+) + f(s-)) = H[g](s), \quad s \in \mathbb{R}\backslash[-1,1]).$$

Then

$$H[f] = g$$

a.e. in $\mathbb{R}$ and this equation holds pointwise throughout $(-1,1)$. Fix $\varepsilon > 0$. We apply Theorem 1 with $U \equiv 1$ and $S \equiv 0$. We see that for $|x| \leqslant 1-\varepsilon$,

$$\begin{aligned}\left|H\left[\frac{H[f]}{P_n}\right](x)\right| &= \left|\frac{1}{\pi}\int_{\mathbb{R}\backslash[-1,1]} \frac{g(s)}{P_n(s)(s-x)}\,ds\right| \\ &\leqslant \frac{1}{\pi\varepsilon}\int_{\mathbb{R}\backslash[-1,1]} \left|\frac{g}{P_n}\right|(s)\,ds \\ &\leqslant \frac{1}{\pi\varepsilon}||g||_{L_p(\mathbb{R}\backslash[-1,1])}\left|\left|\frac{1}{P_n}\right|\right|_{L_q(\mathbb{R}\backslash[-1,1])}.\end{aligned}$$

So

$$\begin{aligned}&||f - L_n[f]||_{L_\infty[-1+\varepsilon,1-\varepsilon]} \\ &\leqslant \frac{1}{\pi\varepsilon}||P_n||_{L_\infty[-1+\varepsilon,1-\varepsilon]}||g||_{L_p(\mathbb{R}\backslash[-1,1])}\left|\left|\frac{1}{P_n}\right|\right|_{L_q(\mathbb{R}\backslash[-1,1])}.\end{aligned}$$

Now the hypothesis gives the result. □

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